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NEW CENTRAL POLYNOMIALS FOR THE MATRIX ALGEBRA

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ABSTRACT

For $n \geq 3$ we find a central polynomial of degree $(n-1)^2+4$ for the $n \times n$ matrix algebra over a field of characteristic 0. For $n = 3, 4$ our polynomial coincides with the known central polynomials of minimal degree and for $n > 4$ the result gives new central polynomials. Until now, for $n > 4$ the minimal degree of the known central polynomials was n^2.

Introduction

Let $K\langle X\rangle$ be the free associative algebra over a field K of characteristic 0. An element $f(x_1, x_2, \ldots, x_m) \in K\langle X\rangle$ is called a **central polynomial** for the $n \times n$ matrix algebra $M_n(K)$ if $f(r_1, r_2, \ldots, r_m)$ lies in the center of $M_n(K)$ for all $r_1, r_2, \ldots, r_n \in M_n(K)$, and f is not a polynomial identity for $M_n(K)$. The first central polynomials for any n were constructed by Formanek and Razmyslov in [5] and [9] with two different methods. The construction of Formanek yields a central polynomial of degree n^2. The original Razmyslov polynomial was of higher degree but Halpin [7] showed that the method of [9] also gives rise to a central polynomial of degree n^2. For a long time this value was thought to be the minimal value for the degree of central polynomials for $n \times n$ matrices, and this is in fact the case for $n = 1$ and $n = 2$. But the author with Kasparian in [2] and with Piacentini Cattaneo in [3] found central polynomials of degree 8 for

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$M_3(K)$ and of degree 13 for $M_4(K)$, respectively. The main result of this paper is that we construct a central polynomial for $M_n(K)$ of degree $(n-1)^2+4$ for any $n \geq 3$. For $n > 4$ our central polynomial is of degree less than the minimal known degree n^2. For $n = 3$ it is of minimal possible degree as it was shown in [1]. For $n = 4$ we obtain the same central polynomial as in [3] which is of minimal known degree and agrees with the conjecture of Formanek [6] that the minimal degree of the central polynomials for $M_n(K)$ is $(n^2 + 3n - 2)/2$. To obtain the central polynomial of degree $(n-1)^2 + 4$ we give explicitly the following essentially weak polynomial identity for $M_n(K)$:

$$w(x, y_1, \ldots, y_n) = s_{2n-2}(x, x^2, \ldots, x^{n-3}, x^n, y_1, \ldots, y_n)$$
$$+ \sum_{i=1}^{n} x s_{2n-2}(x, x^2, \ldots, x^{n-3}, x^{n-2}, y_1, \ldots, y_i x, \ldots, y_n)$$
$$+ \sum_{1 \leq i < j \leq n} s_{2n-2}(x, x^2, \ldots, x^{n-3}, x^{n-2}, y_1, \ldots, y_i x, \ldots, y_j x, \ldots, y_n),$$

where $s_m(x_1, \ldots, x_m)$ is the standard polynomial of degree m. By the Razmyslov approach in [9] this gives rise to a central polynomial of the right degree. Our essentially weak polynomial identity is a generalization of these found by the author and Rashkova [4] for $n = 3$ and in [3] for $n = 4$. Our starting point was [3] and we follow its exposition. We also refer to [3] for the missing details.

1. Preliminaries

Let K be a field of characteristic zero. We denote by $K\langle X \rangle$ the free associative algebra over K freely generated by a countable set of variables $X = \{x_1, x_2, \ldots\}$ and by $K\langle x_1, \ldots, x_m \rangle$ the subalgebra of rank m. We also use other variables, e.g. $x, y_1, \ldots, y_n$, to denote the free generators. We recall some background.

α) To a polynomial

$$g(t_1, \ldots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \cdots t_{n+1}^{p_{n+1}}$$

in $n + 1$ commuting variables $t_1, \ldots, t_{n+1}$ we associate the polynomial

$$\phi(g) = \phi(g)(x, y_1, \ldots, y_n) = \sum \alpha_p x^{p_1} y_1 x^{p_2} y_2 \cdots x^{p_n} y_n x^{p_{n+1}}$$

from $\in K\langle x, y_1, \ldots, y_n \rangle$. Every $f = f(x, y_1, \ldots, y_n) \in K\langle x, y_1, \ldots, y_n \rangle$ which is multilinear in $y_1, \ldots, y_n$ may be written in the form

$$f = \sum \alpha_{rp} x^{p_1} y_{r_1} x^{p_2} y_{r_2} \cdots x^{p_n} y_{r_n} x^{p_{n+1}} = \sum \phi(g_r)(x, y_{r_1}, \ldots, y_{r_n}).$$

β) Let n be fixed and let e_{ij}, $i,j = 1,\dots,n$, be the matrix units from $M_n(K)$. To any set $\{\bar{y}_q = e_{i_q j_q} \mid q = 1,\dots,n\}$ we relate an oriented graph with n vertices $1,2,\dots,n$ and edges (i_q, j_q), $q = 1,\dots,n$. In order to check if $f(x,y_1,\dots,y_n) = \sum \phi(g_r)(x, y_{r_1},\dots,y_{r_n})$ is a polynomial identity for $M_n(K)$ it is sufficient to calculate $f(\bar{x},\bar{y}_1,\dots,\bar{y}_n)$ for $\bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn}$, where $\rho_1,\dots,\rho_n$ are commuting variables and $\bar{y}_q = e_{i_q j_q}$, $q = 1,\dots,n$, for all possible (i_q, j_q), $q = 1,\dots,n$. Then

$$\phi(g)(\bar{x}, e_{i_1 j_1},\dots,e_{i_n j_n}) = \delta g(\rho_{i_1},\rho_{i_2},\dots,\rho_{i_n},\rho_{j_n}) e_{i_1 j_n},$$

where δ equals 1 or 0, depending on whether $(i_1,j_1),(i_2,j_2),\dots,(i_n,j_n)$ is or is not a path in the graph.

Definition: A polynomial $f(x_1,\dots,x_m) \in K\langle X\rangle$ is called a **weak polynomial identity** for $M_n(K)$ if $f(x_1,\dots,x_m)$ vanishes when evaluated on all elements of the Lie algebra sl_n of all traceless matrices of $M_n(K)$. If the weak polynomial identity is not a polynomial identity for $M_n(K)$ it is called an **essentially weak polynomial identity**. ∎

γ) Let the polynomial $f(x,y_1,\dots,y_n) \in K\langle x,y_1,\dots,y_n\rangle$ be multilinear in $y_1,\dots,y_n$. In order to prove that $f(x,y_1,\dots,y_n)$ vanishes for all $\bar{x} \in sl_n$ and all $\bar{y}_1,\dots,\bar{y}_n \in M_n(K)$ it is sufficient to consider $\bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn}$, where $\rho_1,\dots,\rho_n$ are commuting variables satisfying $\rho_1 + \cdots + \rho_n = 0$ and $\bar{y}_q = e_{i_q j_q}$, $q = 1,\dots,n$. Let us assume that $f(x,y_1,\dots,y_n) = \sum \phi(g_r)(x,y_{r_1},\dots,y_{r_n})$ is a polynomial identity for $M_{n-1}(K)$. If $f(\bar{x},\bar{y}_1,\dots,\bar{y}_n) \neq 0$ then the graph related to $\bar{y}_q = e_{i_q j_q}$, $q = 1,\dots,n$, contains a path $(i_{r_1}, j_{r_1}),\dots,(i_{r_n}, j_{r_n})$ going through all the n vertices. Up to a permutation of the indices $1,\dots,n$, this is one of the paths $(1,2),\dots,(i,i+1),\dots,(j-1,j),(j,i),(i,j+1),\dots,(n-1,n)$, where $i \leq j$. Now, if $g_r(t_1,\dots,t_i,t_{i+1},\dots,t_j,t_i,t_{j+1},\dots,t_n)$ is divisible by $t_1 + t_2 + \cdots + t_n$ for all $i \leq j$, this means that $f(\bar{x},\bar{y}_1,\dots,\bar{y}_n)$ vanishes for $\bar{x} \in sl_k$ and $\bar{y}_1,\dots,\bar{y}_n \in M_n(K)$.

2. The weak polynomial identity

The main result of this section is the following.

THEOREM 1: *Let K be a field of characteristic* 0. *The polynomial*

$$w(x,y_1,\dots,y_n)=s_{2n-2}(x,x^2,\dots,x^{n-3},x^n,y_1,\dots,y_n)$$
$$+\sum_{i=1}^{n}xs_{2n-2}(x,x^2,\dots,x^{n-3},x^{n-2},y_1,\dots,y_ix,\dots,y_n)$$
$$+\sum_{1\le i<j\le n}s_{2n-2}(x,x^2,\dots,x^{n-3},x^{n-2},y_1,\dots,y_ix,\dots,y_jx\dots,y_n)$$

from $K\langle x,y_1,\dots,y_n\rangle$ *is an essentially weak polynomial identity for* $M_n(K)$, $n\ge 3$, *and* $w(\bar{x},\bar{y}_1,\dots,\bar{y}_n)=0$ *for all* $\bar{x}\in sl_n$ *and all* $\bar{y}_1,\dots,\bar{y}_n\in M_n(K)$.

The proof of the theorem is based on several lemmas.

LEMMA 1: *Let* $d_1,d_2,\dots,d_{n-2}$ *be positive integers,* $d=(d_1,d_2,\dots,d_{n-2})$ *and let*

$$f_d(x,y_1,\dots,y_n)=s_{2n-2}(x^{d_1},x^{d_2},\dots,x^{d_{n-2}},y_1,\dots,y_n).$$

Let $\bar{x}=\sum_{p=1}^{n}\rho_pe_{pp}$, *where* $\rho_1,\dots,\rho_n$ *are commuting variables,* $\bar{y}_q=e_{i_qj_q}\in M_n(K)$, $q=1,\dots,n$. *Then*

$$f_d(\bar{x},\bar{y}_1,\dots,\bar{y}_n)=\sum_{\sigma\in S_n}(\operatorname{sign}\sigma)g_d(\rho_{\sigma(i_1)},\rho_{\sigma(i_2)},\dots,\rho_{\sigma(i_n)},\rho_{\sigma(j_n)})\bar{y}_{\sigma(1)}\cdots\bar{y}_{\sigma(n)},$$

where $g(t_1,t_2,\dots,t_{n+1})\in K[t_1,t_2,\dots,t_{n+1}]$,

$$g_d(t_1,t_2,\dots,t_{n+1})=\sum_{k_1<\dots<k_{n-2}}\pm\begin{vmatrix}t_{k_1}^{d_1}&t_{k_2}^{d_1}&\dots&t_{k_{n-2}}^{d_1}\\ t_{k_1}^{d_2}&t_{k_2}^{d_2}&\dots&t_{k_{n-2}}^{d_2}\\ & &\dots& \\ t_{k_1}^{d_{n-2}}&t_{k_2}^{d_{n-2}}&\dots&t_{k_{n-2}}^{d_{n-2}}\end{vmatrix}$$

and the sign ±1 *is equal to the sign of the permutation in the summand of* f_d

$$y_1\cdots y_{k_1-1}x^{d_1}y_{k_1}\cdots y_{k_2-1}x^{d_2}\cdots x^{d_{n-3}}y_{k_{n-3}}\cdots y_{k_{n-2}-1}x^{d_{n-2}}y_{k_{n-2}}\cdots y_n.$$

In particular, $g_d(t_1,\dots,t_{n+1})=0$ *if* $d_i=d_j$ *for some* $1\le i<j\le n-2$.

Proof: Let $\bar{x}=\rho_1e_{11}+\dots+\rho_ne_{nn}$ and let $\bar{y}_q$ be matrix units, $q=1,\dots,n$. If $\bar{z}_1=\bar{x}^{d_1},\dots,\bar{z}_{n-2}=\bar{x}^{d_{n-2}},\bar{z}_{n-1}=\bar{y}_1,\dots,\bar{z}_{2n-2}=\bar{y}_n$ then

$$f_d(\bar{x},\bar{y}_1,\dots,\bar{y}_n)=\sum_{\sigma\in S_{2n-2}}(\operatorname{sign}\sigma)\bar{z}_{\sigma(1)}\cdots\bar{z}_{\sigma(2n-2)}.$$

If for $\rho \in S_{2n-2}$ there exists an i such that $\bar{z}_{\rho(i)} = \bar{x}^{d_{p_1}}$ and $\bar{z}_{\rho(i+1)} = \bar{x}^{d_{p_2}}$, then the summand $(\operatorname{sign}\rho)\bar{z}_{\rho(1)}\cdots\bar{z}_{\rho(2n-2)}$ participates in $f_d(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)$ together with $-(\operatorname{sign}\rho)\bar{z}_{\rho(1)}\cdots\bar{z}_{\rho(i-1)}\bar{z}_{\rho(i+1)}\bar{z}_{\rho(i)}\bar{z}_{\rho(i+2)}\cdots\bar{z}_{\rho(2n-2)}$. Since

$$\bar{z}_{\rho(i)}\bar{z}_{\rho(i+1)} - \bar{z}_{\rho(i+1)}\bar{z}_{\rho(i)} = \bar{x}^{d_{p_1}}\bar{x}^{d_{p_2}} - \bar{x}^{d_{p_2}}\bar{x}^{d_{p_1}} = 0,$$

the contribution to $f_d(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)$ is given only by these summands such that no $\bar{x}^{d_{p_1}}$ and $\bar{x}^{d_{p_1}}$ are adjacent. In virtue of β) from Section 1

$$f_d(\bar{x},\bar{y}_1,\ldots,\bar{y}_n) = \sum_{\sigma\in S_n}(\operatorname{sign}\sigma)g_d(\rho_{\sigma(i_1)},\rho_{\sigma(i_2)},\ldots,\rho_{\sigma(i_n)},\rho_{\sigma(j_n)})\bar{y}_{\sigma(1)}\cdots\bar{y}_{\sigma(n)}$$

for some polynomial $g_d(y_1,\ldots,y_{n+1}) \in K[t_1,\ldots,t_{n+1}]$. The polynomial g_d is obtained from

$$\phi(g_d)(x,y_1,\ldots,y_n) = \sum_{k_1<\ldots<k_{n-2}}\sum_{\tau\in S_{n-2}} y_1\cdots y_{k_1-1}x^{d_{\tau(1)}}\times$$

$$\times y_{k_1}\cdots y_{k_2-1}x^{d_{\tau(2)}}\cdots x^{d_{\tau(n-3)}}y_{k_{n-3}}\cdots y_{k_{n-2}-1}x^{d_{\tau(n-2)}}y_{k_{n-2}}\cdots y_n.$$

Hence

$$g_d(t_1,\ldots,t_{n+1}) = \sum_{k_1<\ldots<k_{n-2}} \pm u_k(t_1,\ldots,t_{n+1}),$$

where $k = (k_1,\ldots,k_{n-2})$ and

$$u_k(t_1,\ldots,t_{n+1}) = \sum_{\tau\in S_{n-2}}(\operatorname{sign}\tau)t_{k_1}^{d_{\tau(1)}}\cdots t_{k_{n-2}}^{d_{\tau(n-2)}}.$$

Obviously $u_k(t_1,\ldots,t_{n+1})$ is equal to the determinant of the $(n-2)\times(n-2)$ matrix with entry $t_{k_q}^{d_p}$ in the p-th row and the q-th column. Clearly the sign ± 1 in the sum $\sum \pm u_k(t_1,\ldots,t_{n+1})$ is the same as prescribed in the statement of the lemma. ∎

LEMMA 2: *For the polynomial $g_d(t_1,\ldots,t_{n+1})$ from Lemma 1 and for $1 \le i \le n$*

$$g_d(t_1,\ldots,t_{i-1},0,0,t_{i+2},\ldots,t_{n+1})$$

is equal up to a sign to the determinant

$$\Delta_i = \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ t_1^{d_1} & \ldots & t_{i-1}^{d_1} & t_{i+2}^{d_1} & \ldots & t_{n+1}^{d_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ t_1^{d_{n-2}} & \ldots & t_{i-1}^{d_{n-2}} & t_{i+2}^{d_{n-2}} & \ldots & t_{n+1}^{d_{n-2}} \end{vmatrix}.$$

Proof: By Lemma 1, $g_d(t_1,\dots,t_{i-1},0,0,t_{i+2},\dots,t_{n+1})$ is equal to a linear combination with coefficients ± 1 of the determinants of the $(n-2)\times(n-2)$ matrices $\left(t_{k_q}^{d_p}\right)$ such that i and $i+1$ do not participate between the indices $k_1<\cdots<k_{n-2}$. Hence in the notation of the proof of Lemma 1

$$g_d(t_1,\dots,t_{i-1},0,0,t_{i+2},\dots,t_{n+1})=\sum \pm u_k(t_1,\dots,t_{n+1}),$$

where the summation runs on all $k=(k_1,\dots,k_{n-2})$ such that $1\le k_1<\cdots<k_{n-2}\le n+1$, $k_q\ne i,i+1$, $q=1,\dots,n-2$. The same determinants $u_k(t_1,\dots,t_{n+1})$ participate in the extension by the first row of the $(n-1)\times(n-1)$ determinant Δ_i. The only problem is to show that the signs in both the expressions are the same. For

$$j\ne i,i+1 \quad\text{and}\quad k=(1,\dots,i-1,i+2,\dots,j-1,j+1,\dots,n+1)$$

the sign of $\bar{u}_j=u_k$ in $g_d(t_1,\dots,t_{i-1},0,0,t_{i+2},\dots,t_{n+1})$ is equal to the sign of

$$(x^{d_1}y_1)\cdots(x^{d_{i-1}}y_{i-1})y_iy_{i+1}(x^{d_i}y_{i+2})\cdots\times$$
$$\times(x^{d_j-3}y_{j-1})y_j(x^{d_j-3}y_{j+1})\cdots(x^{d_n-3}y_n)x^{d_{n-2}}$$

in the expression of $s_{2n-2}(x^{d_1},\dots,x^{d_{n-2}},y_1,\dots,y_n)$. Obviously for $j>i+1$ (or for $j<i-1$) the signs of $\bar{u}_j$ and $\bar{u}_{j+1}$ in both g_d and Δ_i are opposite, the same holds for the signs of $\bar{u}_{i-1}$ and $\bar{u}_{i+2}$. Hence

$$\begin{aligned}\Delta_i&=\bar{u}_1-\bar{u}_2+\cdots\pm\bar{u}_{i-1}\mp\bar{u}_{i+2}\pm\cdots\pm\bar{u}_{n+1}\\&=\pm g_d(t_1,\dots,t_{i-1},0,0,t_{i+2},\dots,t_{n+1}).\end{aligned}\qquad\blacksquare$$

LEMMA 3: *For $g_d(t_1,\dots,t_{n+1})$ from Lemma 1 and $1\le i\le n$*

$$g_d(t_1,\dots,t_{n+1})+g_d(t_1,\dots,t_{i-1},t_{i+1},t_i,t_{i+2},\dots,t_{n+1})$$
$$=2g_d(t_1,\dots,t_{i-1},0,0,t_{i+2},\dots,t_{n+1}).$$

Therefore

$$g_d(t_1,\dots,t_{i-1},t_i,t_i,t_{i+1},\dots,t_n)=g_d(t_1,\dots,t_{i-1},0,0,t_{i+1},\dots,t_n).$$

Proof: We consider the case $i=1$; the proof for i arbitrary is similar. By Lemma 1

$$g_d(t_1,t_2,t_3,\dots,t_{n+1})=$$

$$\sum \pm \begin{vmatrix} t_1^{d_1} & t_2^{d_1} & t_{k_3}^{d_1} & \dots & t_{k_{n-2}}^{d_1} \\ & & \dots & & \\ t_1^{d_{n-2}} & t_2^{d_{n-2}} & t_{k_3}^{d_{n-2}} & \dots & t_{k_{n-2}}^{d_{n-2}} \end{vmatrix} + \sum_{k_1>2} \pm \begin{vmatrix} t_{k_1}^{d_1} & \dots & t_{k_{n-2}}^{d_1} \\ & \dots & \\ t_{k_1}^{d_{n-2}} & \dots & t_{k_{n-2}}^{d_{n-2}} \end{vmatrix}$$
$$+ \sum_{k_2>2} \pm \left(\begin{vmatrix} t_1^{d_1} & t_{k_2}^{d_1} & \dots & t_{k_{n-2}}^{d_1} \\ & \dots & & \\ t_1^{d_{n-2}} & t_{k_2}^{d_{n-2}} & \dots & t_{k_{n-2}}^{d_{n-2}} \end{vmatrix} - \begin{vmatrix} t_2^{d_1} & t_{k_2}^{d_1} & \dots & t_{k_{n-2}}^{d_1} \\ & \dots & & \\ t_2^{d_{n-2}} & t_{k_2}^{d_{n-2}} & \dots & t_{k_{n-2}}^{d_{n-2}} \end{vmatrix} \right).$$

Using the elementary properties of determinants we obtain that

$$g_d(t_1, t_2, t_3, \dots, t_{n+1}) + g_d(t_2, t_1, t_3, \dots, t_{n+1}) = 2g_d(0, 0, t_3, \dots, t_{n+1}).$$

The second statement of the lemma follows immediately from the first by replacing t_{i+1} with t_i. ∎

LEMMA 4: *Let $h_l(t_1, \dots, t_m)$, $m > 0$, be the complete symmetric function of degree $l \geq 0$ in the commuting variables $t_1, \dots, t_m$, $h_0 = 1$. For $r \geq 0$ we define an $m \times m$ matrix*

$$D_{mr} = D_{mr}(t_1, \dots, t_m) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \\ t_1^2 & t_2^2 & \dots & t_m^2 \\ & & \dots & \\ t_1^{m-2} & t_2^{m-2} & \dots & t_m^{m-2} \\ t_1^{m-1+r} & t_2^{m-1+r} & \dots & t_m^{m-1+r} \end{pmatrix}.$$

Then

$$\det(D_{mr}) = h_r(t_1, \dots, t_m) \prod_{1 \leq p < q \leq m} (t_q - t_p).$$

Proof: We use the notation of [8, pp. 23 – 24]. For $\alpha = (\alpha_1, \dots, \alpha_m)$ we define a polynomial $a_\alpha = a_\alpha(t_1, \dots, t_m)$ in commuting variables obtained by antisymmetrizing of $t_1^{\alpha_1} \dots t_m^{\alpha_m}$, i.e.

$$a_\alpha = \sum_{\sigma \in S_m} (\operatorname{sign} \sigma) t_{\sigma(1)}^{\alpha_1} \cdots t_{\sigma(m)}^{\alpha_m}.$$

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition in not more than m parts, i.e. $\lambda_1 \geq \dots \geq \lambda_m \geq 0$, and let $\delta = (m-1, m-2, \dots, 1, 0)$. Then $a_{\lambda+\delta}$ can be written as a determinant

$$a_{\lambda+\delta} = \det(t_i^{\lambda_j + m - j})$$

and the Schur function $s_\lambda(t_1, \dots, t_m)$ related with λ is equal to $a_{\lambda+\delta}/a_\delta$. Clearly, for $\lambda = (r)$

$$D_{mr} = (-1)^{m(m-1)/2} a_{\lambda+\delta}, \quad D_{m0} = (-1)^{m(m-1)/2} a_\delta = \prod_{1 \leq p < q \leq m} (t_q - t_p).$$

This gives immediately the proof of the lemma because

$$s_{(r)}(t_1,\ldots,t_m)=h_r(t_1,\ldots,t_m). \qquad \blacksquare$$

The following lemma is an immediate consequence of Lemma 1.

LEMMA 5: *Let $c_1,\ldots,c_n,d_0\geq 0$ and let $d_1,\ldots,d_{n-2}\geq 1$, $d=(d_1,\ldots,d_{n-2})$. Then in the notation of Lemma 1 for $\bar{x}=\rho_1e_{11}+\cdots+\rho_ne_{nn}$, $\bar{y}_{i_qj_q}$, $q=1,\ldots,n$,*

$$\bar{x}^{d_0}s_{2n-2}(\bar{x}^{d_1},\ldots,\bar{x}^{d_{n-2}},\bar{y}_1\bar{x}^{c_1},\ldots,\bar{y}_n\bar{x}^{c_n})$$

$$=\sum_{\sigma\in S_n}(\operatorname{sign}\sigma)g_d(\rho_{\sigma(i_1)},\ldots,\rho_{\sigma(i_n)},\rho_{\sigma(j_n)})\times$$
$$\times\rho_{\sigma(i_1)}^{d_0}\rho_{\sigma(i_2)}^{c_{\sigma(i_1)}}\cdots\rho_{\sigma(i_n)}^{c_{\sigma(i_{n-1})}}\rho_{\sigma(j_n)}^{c_{\sigma(j_n)}}\bar{y}_{\sigma(1)}\cdots\bar{y}_{\sigma(n)}.$$

LEMMA 6: *For*

$$w(x,y_1,\ldots,y_n)=s_{2n-2}(x,x^2,\ldots,x^{n-3},x^n,y_1,\ldots,y_n)$$

$$+\sum_{i=1}^{n}xs_{2n-2}(x,x^2,\ldots,x^{n-3},x^{n-2},y_1,\ldots,y_ix,\ldots,y_n)$$
$$+\sum_{1\leq i<j\leq n}s_{2n-2}(x,x^2,\ldots,x^{n-3},x^{n-2},y_1,\ldots,y_ix,\ldots,y_jx\ldots,y_n)$$

and $\bar{x}=\rho_1e_{11}+\cdots+\rho_ne_{nn}$, $\bar{y}_q=e_{i_qj_q}$, $q=1,\ldots,n$, the equality

$$w(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)=\sum_{\sigma\in S_n}(\operatorname{sign}\sigma)g(\rho_{\sigma(i_1)},\ldots,\rho_{\sigma(i_n)},\rho_{\sigma(j_n)})\bar{y}_{\sigma(1)}\cdots\bar{y}_{\sigma(n)}$$

holds where

$$g(t_1,\ldots,t_{n+1})=$$

$$g_{(1,2,\ldots,n-3,n)}(t_1,\ldots,t_{n+1})+g_{(1,2,\ldots,n-3,n-2)}(t_1,\ldots,t_{n+1})e_2(t_1,\ldots,t_{n+1})$$

and $e_2(t_1,\ldots,t_{n+1})$ is the second elementary symmetric function in $t_1,\ldots,t_{n+1}$.

Proof: By Lemma 5

$$w(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)=\sum_{\sigma\in S_n}(\operatorname{sign}\sigma)g(\rho_{\sigma(i_1)},\ldots,\rho_{\sigma(i_n)},\rho_{\sigma(j_n)})\bar{y}_{\sigma(1)}\cdots\bar{y}_{\sigma(n)},$$

where

$$g(t_1,\dots,t_{n+1}) = g_{(1,2,\dots,n-3,n)}(t_1,\dots,t_{n+1}) + g_{(1,2,\dots,n-3,n-2)}(t_1,\dots,t_{n+1}) \left(t_1 \sum_{q=2}^{n+1} t_q + \sum_{2\le p<q\le n+1} t_p t_q \right)$$

and this completes the proof because

$$t_1 \sum_{q=2}^{n+1} t_q + \sum_{2\le p<q\le n+1} t_p t_q = e_2(t_1,\dots,t_{n+1}). \qquad \blacksquare$$

Proof of Theorem 1: By the Amitsur–Levitzki theorem, $s_{2n-2}(x_1,\dots,x_{2n-2})$ is a polynomial identity for $M_{n-1}(K)$. In order to prove that $w(x,y_1,\dots,y_n)$ vanishes for all $\bar{x} \in sl_n$ and all $\bar{y}_1,\dots,\bar{y}_n \in M_n(K)$, it is sufficient, by γ) of Section 1, to establish that $w(\bar{x},\bar{y}_1,\dots,\bar{y}_n) = 0$ for $\bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn}$, with $\rho_1 + \cdots + \rho_n = 0$ and

$$\bar{y}_1 = e_{12}, \quad \dots, \quad \bar{y}_{j-1} = e_{j-1,j}, \quad \bar{y}_j = e_{ji},$$

$$\bar{y}_{j+1} = e_{i,j+1}, \quad \bar{y}_{j+2} = e_{j+1,j+2}, \quad \dots, \quad \bar{y}_n = e_{n-1,n}$$

for all $i \le j$. In virtue of Lemma 6, $w(\bar{x},\bar{y}_1,\dots,\bar{y}_n) = 0$ if

$$g(t_1,\dots,t_{n+1}) = g_{(1,2,\dots,n-3,n)}(t_1,\dots,t_{n+1}) + g_{(1,2,\dots,n-3,n-2)}(t_1,\dots,t_{n+1}) e_2(t_1,\dots,t_{n+1})$$

satisfies the property that $t_1 + \cdots + t_n$ divides $g(t_1,\dots,t_{j-1},t_i,t_{j+1},\dots,t_n)$ for all $1 \le i < j \le n+1$.

Let $v_r(t_1,\dots,t_{n+1}) = g_{(1,\dots,n-3,n-2+r)}(t_1,\dots,t_{n+1})$. Then

$$g(t_1,\dots,t_{n+1}) = v_2(t_1,\dots,t_{n+1}) + v_0(t_1,\dots,t_{n+1}) e_2(t_1,\dots,t_{n+1}).$$

We apply induction on the difference $j - i$. For simplicity of the notation we consider the case $i = 1$ only; the general case is similar. First, let $j = i + 1$. By Lemmas 2 and 3

$$v_r(t_1,t_1,t_2,\dots,t_n) = v_r(0,0,t_2,\dots,t_n).$$

Applying Lemma 4 we obtain that

$$v_2(0,0,t_2,\dots,t_n) = \pm h_2(t_2,\dots,t_n) \prod_{1<p<q\le n} (t_q - t_p),$$

$$v_0(0,0,t_2,\ldots,t_n)=\pm\prod_{1<p<q\leq n}(t_q-t_p),$$

with the same sign ± 1. Therefore

$$g(t_1,t_1,t_2,\ldots,t_n)=\pm\left(h_2(t_2,\ldots,t_n)+e_2(t_1,t_1,t_2,\ldots,t_n)\right)\prod_{1<p<q\leq n}(t_q-t_p)$$

and easy calculations show that

$$h_2(t_2,\ldots,t_n)+e_2(t_1,t_1,t_2,\ldots,t_n)=(t_1+\cdots+t_n)^2.$$

Hence $g(t_1,t_1,t_2,\ldots,t_n)$ is divisible by $t_1+\cdots+t_n$.

Now, let $j-i>1$. By the inductive assumption

$$g(t_1,\ldots,t_{j-1},t_1,t_j,\ldots,t_n)=(t_1+\cdots+t_n)u(t_1,\ldots,t_n)$$

for some $u\in K[t_1,\ldots,t_n]$. By Lemma 3

$$v_r(t_1,\ldots,t_{j-1},t_j,t_1,t_{j+1},\ldots,t_n)=$$

$$-v_r(t_1,\ldots,t_{j-1},t_1,t_j,t_{j+1},\ldots,t_n)+2v_r(t_1,\ldots,t_{j-1},0,0,t_{j+1},\ldots,t_n)$$

and we obtain that

$$\begin{aligned}
&g(t_1,\ldots,t_{j-1},t_j,t_1,t_{j+1},\ldots,t_n)\\
&=-(v_2(t_1,\ldots,t_{j-1},t_1,t_j,t_{j+1},\ldots,t_n)\\
&+v_0(t_1,\ldots,t_{j-1},t_1,t_j,t_{j+1},\ldots,t_n)e_2(t_1,\ldots,t_j,t_1,t_{j+1},\ldots,t_n))\\
&+2(v_2(t_1,\ldots,t_{j-1},0,0,t_{j+1},\ldots,t_n)\\
&+v_0(t_1,\ldots,t_{j-1},0,0,t_{j+1},\ldots,t_n)e_2(t_1,\ldots,t_j,t_1,t_{j+1},\ldots,t_n))\\
&=(t_1+\ldots+t_n)u(t_1,\ldots,t_n)\\
&\pm 2\left(h_2(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)+e_2(t_1,t_1,t_2,\ldots,t_n)\right)\prod_{\substack{1\leq p<q\leq n\\ p,q\neq j}}(t_q-t_p).
\end{aligned}$$

Obviously, for $j>1$

$$\begin{aligned}
&h_2(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)+e_2(t_1,t_1,t_2,\ldots,t_n)\\
&=(h_2(t_1,\ldots,t_n)-t_j(t_1+\cdots+t_n))+(t_1(t_1+\cdots+t_n)+e_2(t_1,\ldots,t_n))\\
&=(t_1+\cdots+t_n)^2+(t_1-t_j)(t_1+\cdots+t_n)
\end{aligned}$$

and $g(t_1,\ldots,t_j,t_1,t_{j+1},\ldots,t_n)$ is divisible by $t_1+\ldots+t_n$.

Finally, for

$$\bar{x}=\rho_1 e_{11}+\cdots+\rho_n e_{nn},\quad \bar{y}_1=e_{11},\quad \bar{y}_2=e_{12},\quad \bar{y}_3=e_{23},\quad \ldots,\quad \bar{y}_n=e_{n-1,n},$$

$$\begin{aligned}w(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)&=g(\rho_1,\rho_1,\rho_2,\ldots,\rho_n)e_{1n}\\&=(\rho_1+\cdots+\rho_n)\prod_{2\le p<q\le n}(\rho_q-\rho_p)e_{1n}.\end{aligned}$$

If we choose $\rho_1,\ldots,\rho_n$ pairwise different and such that $\rho_1+\cdots+\rho_n\neq 0$, we obtain that $w(\bar{x},\bar{y}_1,\ldots,\bar{y}_n)\neq 0$. Hence $w(x,y_1,\ldots,y_n)$ is not a polynomial identity for $M_n(K)$ and this completes the proof of the theorem. ∎

3. The central polynomial

Let $[u,v]=uv-vu$ be the commutator of u and v. We make use of the following result by Razmyslov [9].

LEMMA 7: *Let $f(x_1,\ldots,x_m)$ be a multilinear essentially weak polynomial identity for $M_n(K)$ such that $f([x_1,x_{m+1}],x_2,\ldots,x_m)$ is an ordinary polynomial identity for $M_n(K)$. Let us express $f(x_1,\ldots,x_m)$ in the form*

$$f(x_1,\ldots,x_m)=\sum \alpha_{pq}p(x_2,\ldots,x_m)x_1q(x_2,\ldots,x_m),$$

where p and q are monomials not depending on x_1. Then

$$f^*(x_1,\ldots,x_m)=\sum \alpha_{pq}q(x_2,\ldots,x_m)x_1p(x_2,\ldots,x_m)$$

is a central polynomial for $M_n(K)$.

By a complete linearization in x of a homogeneous of degree k in x polynomial $w(x,y_1,\ldots,y_n)\in K\langle x,y_1,\ldots,y_n\rangle$ we mean the multilinear in $x_1,\ldots,x_k$ component $w'(x_1,\ldots,x_k,y_1,\ldots,y_n)$ of the polynomial $w(x_1+\cdots+x_k,y_1,\ldots,y_n)$. Now we prove the main result of our paper.

THEOREM 2: *Let $w'(x_1,\ldots,x_k,y_1,\ldots,y_n)$ be the complete linearization in x of the essentially weak polynomial identity for $M_n(K)$*

$$w(x,y_1,\ldots,y_n)=s_{2n-2}(x,x^2,\ldots,x^{n-3},x^n,y_1,\ldots,y_n)$$

$$+\sum_{i=1}^{n} x s_{2n-2}(x, x^2, \ldots, x^{n-3}, x^{n-2}, y_1, \ldots, y_i x, \ldots, y_n)$$
$$+\sum_{1\le i<j\le n} s_{2n-2}(x, x^2, \ldots, x^{n-3}, x^{n-2}, y_1, \ldots, y_i x, \ldots, y_j x, \ldots, y_n),$$

$k = (n^2 - 3n + 6)/2$, and let

$$f(x_1, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k) = w'(x_1, [x_2, z_2], \ldots, [x_k, z_k], y_1, \ldots, y_n).$$

Then

$$f^*(x_1, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k)$$

is a central polynomial of degree $(n-1)^2 + 4$ for $M_n(K)$, $k \ge 3$.

Proof: Obviously the polynomial $f(x_1, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k)$ is of degree $(n-1)^2 + 4$. By Lemma 7 it is sufficient to show that

$$f([x_1, z_1], x_2, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k)$$

is an ordinary polynomial identity for $M_n(K)$ and $f(x_1, \ldots, x_k, y_1, \ldots, z_k)$ is not.

By Theorem 1, $w(x, y_1, \ldots, y_n)$ vanishes for $\bar{x} \in sl_n$, $\bar{y}_r \in M_n(K)$, $r = 1, \ldots, n$. Hence its linearization $w'(x_1, \ldots, x_k, y_1, \ldots, y_n)$ vanishes for $\bar{x}_h \in sl_n$, $h = 1, \ldots, k$, $\bar{y}_r \in M_n(K)$, $i = 1, \ldots, n$. Since the commutators $[\bar{x}_h, \bar{z}_h]$ belong to sl_n for $\bar{x}_h, \bar{z}_h \in M_n(K)$, we obtain that $f([x_1, z_1], x_2, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k)$ is a polynomial identity for $M_n(K)$.

Any diagonal matrix $\bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn} \in sl_n$ can be written as a commutator of two matrices. Hence we shall show that $f(x_1, \ldots, x_k, y_1, \ldots, y_n, z_2, \ldots, z_k)$ is not a polynomial identity for $M_n(K)$ if we establish that

$$\tilde{w}(x, y_1, \ldots, y_n) = w'(1, \underbrace{x, \ldots, x}_{k-1}, y_1, \ldots, y_n)$$

is not a weak polynomial identity for $M_n(K)$. Obviously, up to a multiplicative constant, $\tilde{w}(x, y_1, \ldots, y_n)$ is equal to the homogeneous component of $w(1 + x, y_1, \ldots, y_n)$ of degree $k - 1$ in x. We shall calculate $\tilde{w}$ for

$$\bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn},$$

$$\bar{y}_1 = e_{12}, \quad \bar{y}_2 = e_{21}, \quad \bar{y}_3 = e_{13}, \quad \bar{y}_4 = e_{34}, \quad \bar{y}_5 = e_{45}, \quad \ldots, \quad \bar{y}_n = e_{n-1,n}.$$

We use the notation of Section 2. As in the proof of Theorem 1 it is easy to see that

$$\tilde{w}(\bar{x},\bar{y}_1,\dots,\bar{y}_n)=g'(\rho_1,\rho_2,\rho_1,\rho_3,\rho_4,\dots,\rho_n)e_{1n},$$

where $g(t_1,\dots,t_{n+1})$ is the polynomial from Lemma 6 and $g'(t_1,\dots,t_{n+1})$ is the homogeneous component of degree $k-1$ of $g(1+t_1,\dots,1+t_{n+1})$. Using Lemma 1 it is easy to see that

$$\begin{aligned} g'(t_1,\dots,t_{n+1}) =&ng_{(1,2,\dots,n-3,n-1)}(t_1,\dots,t_{n+1})\\ &+ng_{(1,2,\dots,n-3,n-2)}(t_1,\dots,t_{n+1})h_1(t_1,\dots,t_{n+1})\\ =&n(v_1(t_1,\dots,t_{n+1})+v_0(t_1,\dots,t_{n+1})h_1(t_1,\dots,t_{n+1})). \end{aligned}$$

Applying Lemmas 2 – 4 we obtain that for $r\geq 0$,

$$\begin{aligned} v_r(t_1,t_2,t_1,t_3,t_4,\dots,t_n) =& -v_r(t_1,t_1,t_2,t_3,\dots,t_n)+2v_r(t_1,0,0,t_3,\dots,t_n)\\ =& -v_r(0,0,t_2,\dots,t_n)+2v_r(t_1,0,0,t_3,\dots,t_n)\\ =& \pm h_r(t_2,\dots,t_n)\\ &\prod_{2\leq i<j\leq n}(t_j-t_i)\pm 2h_r(t_1,t_3,t_4,\dots,t_n)\\ &\prod_{\substack{1\leq i<j\leq n\\ i,j\neq 2}}(t_j-t_i). \end{aligned}$$

Hence

$$\begin{aligned} &g'(t_1,t_2,t_1,t_3,t_4,\dots,t_n)=nv_1(t_1,t_2,t_1,t_3,t_4,\dots,t_n)\\ &+nv_0(t_1,t_2,t_1,t_3,t_4,\dots,t_n)(2t_1+t_2+t_3+\ldots+t_n)\\ =&\pm n((t_2+t_3+\ldots+t_n)+(2t_1+t_2+t_3+\ldots+t_n))\prod_{2\leq i<j\leq n}(t_j-t_i)\\ &\pm 2n((t_1+t_3+t_4+\ldots+t_n)+(2t_1+t_2+t_3+\ldots+t_n))\prod_{\substack{1\leq i<j\leq n\\ i,j\neq 2}}(t_j-t_i)\\ =&2n(t_1+\ldots+t_n)\left(\pm\prod_{2\leq i<j\leq n}(t_j-t_i)\pm\prod_{\substack{1\leq i<j\leq n\\ i,j\neq 2}}(t_j-t_i)\right)\\ &\pm 2n(t_1-t_2)\prod_{\substack{1\leq i<j\leq n\\ i,j\neq 2}}(t_j-t_i). \end{aligned}$$

Now, if we choose pairwise different $\rho_1,\dots,\rho_n$ such that $\rho_1+\cdots+\rho_n=0$ we obtain that

$$v'(\rho_1,\rho_2,\rho_1,\rho_3,\rho_4,\dots,\rho_n)\neq 0.$$

This means that $\tilde{w}(x, y_1, \ldots, y_n)$ is not a weak polynomial identity for $M_n(K)$ and this completes the proof of the theorem. ∎

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